Chapter 9. First-Order Differential Equations

9.2 First-Order Linear Equations

Definition. A first-order differential equation that **can be** written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are functions of x, is a *linear* first-order equation and the above equation is the *standard form* of the D.E.

Theorem. The solution of the linear equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is

$$y \in \frac{1}{v(x)} \int v(x)Q(x)\,dx,$$

where

$$v(x) \in e^{\int P(x)\,dx}.$$

In the "formula" for $v(x)$, we can simply set v equal to the exponentiation of ANY antiderivative of P.

Proof. First, we multiply both sides of the equation by some function v (called an *integrating factor*) which will transform the left-hand side of the D.E. into the derivative of the product $v(x)y$ (this is a constraint on v that we will deal with shortly):

$$\begin{aligned}
\frac{dy}{dx} + P(x) &= Q(x) \\
v(x)\frac{dy}{dx} + P(x)v(x)y &= v(x)Q(x) \\
\frac{d}{dx}\left[v(x)\cdot y\right] &= v(x)Q(x) \\
v(x)\cdot y &\in \int v(x)Q(x)\,dx \\
y &\in \frac{1}{v(x)}\int v(x)Q(x)\,dx.
\end{aligned}$$

Now we must deal with the constraint on v:

$$\begin{aligned}
\frac{d}{dx}\left[v\cdot y\right] &= v\frac{dy}{dx} + Pvy \\
v\frac{dy}{dx} + y\frac{dy}{dx} &= v\frac{dy}{dx} + Pvy \\
y\frac{dv}{dx} &= Pvy.
\end{aligned}$$

This last equation will hold if

$$\begin{aligned}
\frac{dy}{dx} &= Pv \\
\frac{dv}{v} &= P\,dx \\
\int \frac{dv}{v} &\in \int P\,dx
\end{aligned}$$

$$
\begin{aligned}
\ln v &\in \int P\,dx \\
&\qquad \text{(notice } v > 0 \text{ by hypothesis)} \\
e^{\ln v} &\in e^{\int P\,dx} \\
v &\in e^{\int P\,dx}.
\end{aligned}
$$

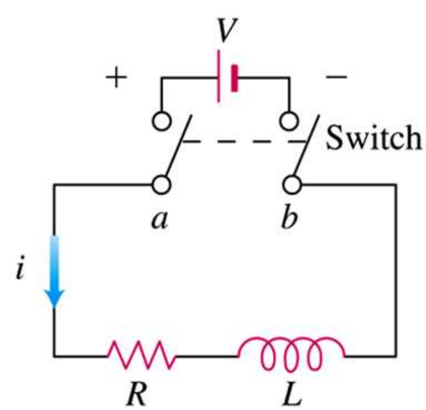
Q.E.D.

Example. Page 526 numbers 8 and 16.

Note. The diagram below represents an electrical circuit whose total resistance is a constant R ohms and whose self-inductance, shown as a coil, is L henries, also a constant. There is a switch whose terminal at a and b can be closed to connect a constant electrical source of V volts. Ohm's Law, $V = RI$, has to be modified for such a circuit. The modified form is

$$L\frac{di}{dt} + Ri = V,$$

where i is the intensity of the current in amperes and t is the time in seconds. By solving this equation, we can predict how the current i will flow after the switch is closed.



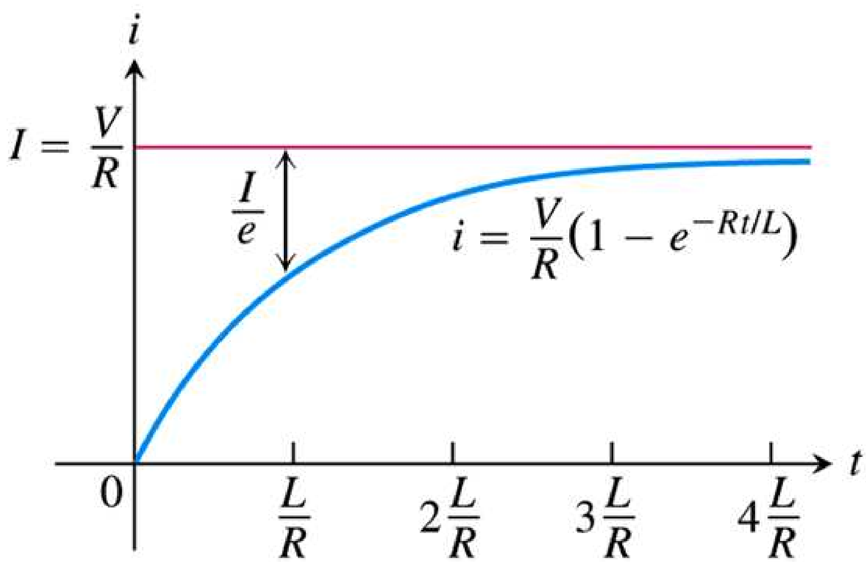
Page 526 Figure 9.8

Example. Page 527 number 28. Solution: $i = \dfrac{V}{R} - \dfrac{V}{R}e^{-(R/L)t}$.

Note. In the above problem, we have

$$\lim_{t\to\infty} i = \lim_{t\to\infty}\left(\frac{V}{R}-\frac{V}{R}e^{-(R/L)t}\right)=\frac{V}{R}-\frac{V}{R}\cdot 0=\frac{V}{R}.$$

From the graph of the solution, we see why $i = V/R$ is called a *steady-state value*. In fact, the solution is expressed as the sum of a steady state solution V/R and a *transient solution* $-(V/R)e^{-(R/L)t}$.



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