Chapter 6. Duality

6.1. Examples of Dual Spaces

Note. Recall that the dual space of a normed linear space X is the space of all bounded linear functionals from X to the scalar field $\mathbb{F}$, originally denoted $\mathcal{B}(X, \mathbb{F})$, but more often denoted X^*. In this section we find the duals of the ℓ^p spaces for $1 \le p < \infty$ and L^p for $1 \le p < \infty$. Our lack of background in Lebesgue measure and integration does not allow us to give totally rigorous proofs for some of these results. In the following result, the verb "is" means that there is a surjective isometry between the two relevant spaces.

Theorem 6.1.

(a) The dual of c_0 (the space of all sequences which converge to 0, with the sup norm) is ℓ^1.

(b) The dual of ℓ^1 is ℓ^∞.

(c) The dual of ℓ^p for $1 < p < \infty$ is ℓ^q where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Note. Our text is a little unclear (in my opinion) on some of the details of the proof of Theorem 6.1. Therefore, we present a proof that follows the technique of Reed and Simon's *Functional Analysis* (see pages 73 and 74), but use the notation of Promislow.

Proof of Theorem 6.1(a). The dual of c_0 is ℓ^1.

(1) Let $f = (f(1), f(2), \ldots) \in c_0$ and let $g = (g(1), g(2), \ldots) \in \ell^1$. Then we claim the mapping $\phi_g(f) = \sum_{k=1}^{\infty} f(k)g(k)$ is a bounded linear functional on c_0. By Hölder's Inequality for ℓ^p $(1 \leq p < \infty)$ we have (since $c_0 \subseteq \ell^\infty$, we use the sup norm on c_0)

$$\sum_{k=1}^{\infty} |f(k)g(k)| \leq \|g\|_1 \|f\|_\infty \qquad (*)$$

so the series converges absolutely and for $f_1, f_2 \in c_0$ we have

$$\phi_g(f_1 + f_2) = \sum_{k=1}^{\infty} (f_1 + f_2)(k)g(k) = \sum_{k=1}^{\infty} (f_1(k) + f_2(k))g(k)$$

$$= \sum_{k=1}^{\infty} f_1(k)g(k) + \sum_{k=1}^{\infty} f_2(k)g(k) = \phi_g(f_1) + \phi_g(f_2)$$

and for any $\alpha \in \mathbb{F}$, we have

$$\phi_g(\alpha f) = \sum_{k=1}^{\infty} (\alpha f)(k)g(k) = \sum_{k=1}^{\infty} \alpha f(k)g(k) = \alpha \sum_{k=1}^{\infty} f(k)g(k) = \alpha \phi_g(f).$$

So $\phi_g : c_0 \to \mathbb{F}$ is linear. From $(*)$,

$$|\phi_g(f)| = \left| \sum_{k=1}^{\infty} f(k)g(k) \right| \leq \sum_{k=1}^{\infty} |f(k)g(k)| \leq \|g\|_1 \|f\|_\infty,$$

so

$$\|\phi_g\| = \sup\{|\phi_g(f)| \mid f \in c_0, \|f\|_\infty = 1\} \leq \|g\|_1 < \infty,$$

so ϕ_g is a bounded linear functional.

(2) Now we show that all bounded linear functionals are of the form ϕ_g for some $g \in \ell^1$. Let ϕ be a bounded linear functional on c_0. Define $\delta_i \in c_0$ as the sequence with ith entry 1 and all other entries 0. For each $k \in \mathbb{N}$, define $g(k) = \phi(\delta_k)$ and define

$$h(k) = \begin{cases} |g(k)|/g(k) & \text{if } g(k) \neq 0 \\ 0 & \text{if } g(k) = 0. \end{cases}$$

For fixed $N \in \mathbb{N}$, let $h_N = \sum_{k=1}^{N} h(k)\delta_k$ (recall $\delta_k \in c_0$ and $h(k) \in \mathbb{F}$). Then $h_N \in c_0$ and $\|h_N\|_\infty = 1$. Then

$$\begin{aligned} \phi(h_N) &= \phi\left(\sum_{k=1}^{N} h(k)\delta_k\right) = \sum_{k=1}^{N} \phi(h(k)\delta_k) \\ &= \sum_{k=1}^{N} h(k)\phi(\delta_k) \\ &= \sum_{k=1}^{N} h(k)g(k) \text{ since } g(k) = \phi(\delta_k) \\ &= \sum_{k=1}^{N} |g(k)| \text{ by definition of } h(k). \end{aligned}$$

Now $|\phi(h_N)| \leq \|h_N\|_\infty \|\phi\|_{c_0^*}$ by the definition of operator norm in c_0^*. So

$$\begin{aligned} \sum_{k=1}^{N} |g(k)| &= \sum_{k=1}^{N} |\phi(\delta_k)| \text{ by definition of } g(k) \\ &\leq \|\phi\|_{c_o^*} \text{ since } \|h_N\|_\infty = 1. \end{aligned}$$

Now the right hand side depends only on ϕ and $N \in \mathbb{N}$ on the left had side is arbitrary. So

$$\sum_{k=1}^{\infty} |g(k)| = \sum_{k=1}^{\infty} |\phi(\delta_k)| \leq \|\phi\|_{c_o^*}, \qquad (**)$$

and $g \in \ell^1$.

(3) Now we want to show that ϕ given as an element of c_0^* is the same as ϕ_g given in Part (1) of the proof where g is defined in Part (2). By definition, for any $f \in c_{00}$ (the subset of c_0 of all sequences with only a finite number of nonzero entries), $\phi(f) = \phi_g(f)$. By Example 2.18, c_{00} is a dense subset of c_0. Since ϕ is

continuous (it is bounded by definition; apply Theorem 2.6) and ϕ_g is continuous (it is bounded by $\|g\|_1$ as shown above; apply Theorem 2.6) and $\phi = \phi_g$ on dense subset c_{00} in c_0, then $\phi = \phi_g$ on c_0. Therefore, for any $\phi \in c_0^*$, there is $g \in \ell^1$ such that $\phi = \phi_g$. This shows that the mapping $g \to \phi_g$ is surjective (onto).

(4) To see that $g \to \phi_g$ is an isometry, we have that $\|\phi_g\|_{c_0^*} \le \|g\|_1$ from Part (1) (it follows from Hölder's Inequality) and $\|g\|_1 \le \|\phi_g\|_{c_0^*}$ by $(**)$ of Part (2). Therefore $\|g\|_1 = \|\phi_g\|_{c_0^*}$ is an isometry. So there is a surjective isometry from ℓ^1 to c_0^* and so the dual of c_0 is ℓ^1. ∎

Theorem 6.2. If X^* is separable, then X is separable.

Note. The text now discusses signed measures, σ-finite measures, absolutely continuous measures, and the Radon-Nikodym Theorem. We skip these topics here. For a thorough, detailed exploration of these topics, take our Real Analysis 2 class (MATH 5220). These topics are covered in Royden and Fitzpatrick's *Real Anlaysis* 4th Edition, Chapters 17 and 18. I have online notes of these topics at: `http://faculty.etsu.edu/gardnerr/5210/notes3.htm`

Note. We paraphrase a result in the text and take the statement of the result from Royden and Fitzpatrick. For a thorough proof, see Chapter 8 of their *Real Analysis* (page 160) for the result as stated below. For a statement in the more general setting of σ-finite measure spaces (as Promislow gives) see their Chapter 19.

Theorem 6.3′. The Riesz Representation Theorem for the Dual of $L^p(E)$. Let E be a measurable set of real numbers, let $1 \leq p < \infty$, let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. For each $g \in L^q(E)$, define the bounded linear functional $\mathcal{R}_g$ on $L^p(E)$ by

$$\mathcal{R}_g(f) = \int_E gf \text{ for all } f \in L^p(E).$$

Then for each bounded linear functional T on $L^p(E)$, there is a unique function $g \in L^q(E)$ for which $\mathcal{R}_g = T$ and $\|T\|_* = \|g\|_q$.

Note. To simplify the statement of Theorem 6.3, we can say: "The dual space of L^p is L^q where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p < \infty$."

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